5 Counting

5.1 The Basics of Counting

1. two basic principles of counting are the sum rule and the product rule. We present them for two sets, but they both generalize to larger families of sets.

2. the product rule tells us in how many ways you can make one choice from each of two sets of alternatives:

   Theorem 1 *(Product Rule) For any choice of sets A and B, $|A \times B| = |A||B|$.*

3. **Example**: Suppose that you are in a restaurant, and are going to have soup and salad. There are two soups and four salads on the menu. How many choices do you have? By the Product Rule, there are $2 \cdot 4 = 8$ ways to have both soup and salad.

4. the sum rule tells you in how many ways you can make a single choice from two disjoint sets of alternatives (mostly used to add up the solution from different cases that can produce a solution):

   Theorem 2 *(Sum Rule) If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.*

   Although the sum rule tells us that the cardinality of the union of two disjoint sets is the sum of the cardinalities of the two sets, it is typically applied to problems that do not immediately remind us of sets.

5. **Example**: Suppose that you are in a restaurant, and you are going to have either soup or salad but not both. There are two soups and four salads on the menu. How many choices do you have? By the Sum Rule, you have $2 + 4 = 6$ choices.

6. **Example**: We combine the sum and product rules. An example of this application counts the number of passwords adhering to some simple constraints: the length must be at least 5 and at most 7, it must be constructed of uppercase alpha characters and decimal digits, and must contain at least one digit.
   Solution: By the sum rule, the total number P is given by $P = P_5 + P_6 + P_7$, where P_i is the number of legal passwords of length i. But what is P_5? Notice that the number of *illegal* passwords of length 5 is easy to count: there are 26^5 passwords that do not contain any digits (i.e. there ar 26^5 passwords that contain letters only). It follows that P_5 is the total number of passwords on 26 letter s and 10 digits, minus the illgal passwords: $P_5 = 36^5 - 26^5 = 48584800$. This is an example of *indirect* counting: to find the number of ways to perform a task in the presence of constraints, we instead count the number of ways to perform the task with no constraints and subtract from it the number of ways to perform the task while *violating* those constraints. This method is

sometimes easier, and should not be overlooked. We can use the same approach to find $P_6 = 36^6 - 26^6 = 1867866560$ and $P_7 = 36^7 - 26^7 = 70332353920$, and the problem is solved: there are

$$P_5 + P_6 + P_7 = 72248805280$$

acceptable passwords.

7. The basic Principle of Inclusion-Exclusion extends the sum rule to situations in which the two sets of alternatives are not disjoint. The basic instance of the principle applies to unions of two sets.

   - If two sets A and B are disjoint, the cardinality of their union is simply $|A \cup B| = |A| + |B|$ (this is the sum rule).
   - If two sets A and B are not disjoint, say $A \cap B \neq \emptyset$, then

     $$|A \cup B| = |A| + |B| - |A \cap B|,$$

     which is the inclusion-exclusion principle. To see this, note that if $x \in A \cap B$, then x is counted twice in $|A| + |B|$: once in $|A|$ and once in $|B|$. This applies to every element in $A \cap B$, so we must subtract $|A \cap B|$ to correct the overcount.

   Example:

   (a) Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7\}$. Then $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$. By inspection, $|A \cup B| = 7$, but you can also verify that

   $$|A \cup B| = 7 = 5 + 5 - 3 = |A| + |B| - |A \cap B|.$$

   (b) How many positive integers not bigger than 20 are divisible by either 2 or 3? **Solution**: There are $\lfloor 20/2 \rfloor = 10$ that are divisible by 2, and $\lfloor 20/3 \rfloor = 6$ that are divisible by 3. But there are also $\lfloor 20/6 \rfloor = 3$ that are divisible by both 2 and 3, so the total is $10 + 6 - 3 = 13$.

   (c) How many bitstrings of length eight either begin with 00 or end with 101? **Solution**: There are 2^6 that begin with 00, 2^5 that end with 101, and 2^3 that start with 00 and end with 101. So the number of bitstrings with at least one of the two properties is $2^6 + 2^5 - 2^3 = 88$.

   (d) How many bitstrings of length five contain either 11 or 000 (possibly both)? **Solution**: First let's find out how many contain 11. There are 2^3 that begin with 11. Note that we have already counted all those that have the form $111xy$, so in order to count only "new" strings we count those that begin with 011; there are 2^2 of these. Pushing the first occurrence of 11 further to the right, we find that there are 2^2 of the form $x011y$, and finally 2^2 of the form $xy011$. Oops! This last form includes 11011, which we've already counted, so we compensate: there are $2^2 - 1 = 3$ "new" strings of the form $xy011$. Thus there are $2^3 + 2^2 + 2^2 + 3 = 19$ that contain 11. How many contain 000? We take a similar approach. There are $2^2 = 4$ that begin with 000. There are 2 that begin with 1000. Finally, there are 2 of the form $x1000$, for a grand total of $4 + 2 + 2 = 8$. So there are nineteen with 11, and eight with 000. There are also 2 with both substrings, namely 11000 and 00011, so the final tally is $19 + 8 - 2 = 25$ bitstrings with either 11 or 000.

8. The Principle of Inclusion-Exclusion extends to larger collections of sets as well.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

   The principle generalizes to more than three sets.

9. A finite sequence of choices can be represented by a tree diagram, in which the root represents the initial state, leaves represent outcomes, internal vertices represent intermediate states, and edges represent choices. Figure 1 shows a decision tree used to enumerate bitstrings of length three that do not contain 11. The leaves, from left to right, represent the strings 101, 100, 010, 001, and 000. You can see that any branch containing consecutive 1s has been pruned out, leaving only those that do not contain 11. And you can probably guess that the utility of tree diagrams, like that of truth tables and Venn diagrams, is limited to small problems. On the other hand, the tree structure lends itself to computation, so you will probably see this again.

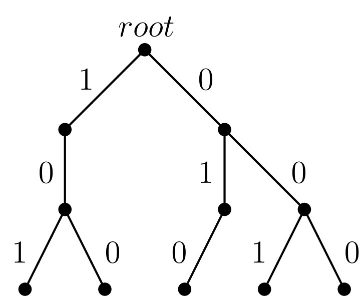

Figure 1: A decision tree for counting 11-free strings

5.2 The Pigeonhole Principle

1. The Pigeonhole Principle says that if $k+1$ pigeons fly into k pigeonholes (or into at most k pigeonholes), at least one pigeonhole must contain at least two pigeons.

Theorem 3 *(The Pigeonhole Principle) If k pigeons occupy $j < k$ pigeonholes, then at least one pigeonhole contains at least two pigeons.*

Example:

(a) The "hello, world" problem for the pigeonhole principle is the "sock problem": In your dresser drawer you have a jumble of socks in two colors, say blue and gray. It's dark, and you don't want to wake your spouse. How many socks must you grab to guarantee that you have a pair of the same color?

Solution: Three socks suffice. You might end up with three blue, or three gray, but with only two colors you're guaranteed to have at least two blue or at least two gray.

(b) Show that in a group of eight people there must be two whose birthdays fall on the same day of the week.

Solution: The pigeons are the people in the group, and the pigeonholes are the days of the week. Since there are 8 people and 7 days, two people must share a day.

The first generalization of the principle is this:

Theorem 4 *If n pigeons occupy k pigeonholes, then at least one pigeonhole contains at least $\lceil n/k \rceil$ pigeons.*

We can use this version to answer more difficult questions: What is the smallest n such that at least one of k boxes must contain at least r of n objects? By Theorem 2, in order to have at least r objects into a box, we need $\lceil n/k \rceil \geq r$
$n/k > r - 1$
$n > k(r-1). Sothesmallestinteger$n$thatforcessomeboxtocontain$r$of$n$objectsis$n=k(r-1)+1.

5.3 Permutations and Combinations

1. recall: for integers $n \geq 0$, the factorial $f(n) = n!$ is defined by

$$n! = \begin{cases} 1, & \text{if } n = 0; \\ (n-1)!n, & \text{if } n > 0. \end{cases}$$

2. a permutation is an ordering, or arrangement, of the elements in a finite set:
Definition: A permutation π of $A = \{a_1, a_2, \ldots, a_n\}$ is an ordering $a_{\pi_1}, a_{\pi_2}, \ldots, a_{\pi_n}$ of the elements of A (no repeats in the list). Example: a permutation of $A = \{1, 2, 3\}$ is $1, 3, 2$

3. there are $n!$ permutations of an n-element set (an n-element set is also called an n-set).

4. an r-permutation of an n-set A ($r \leq n$) is an ordering $a_{\pi_1}, a_{\pi_2}, \ldots, a_{\pi_r}$ of **some** r-subset of A. Example: a 3-permutation of the 4-set $A = \{1, 2, 3, 4\}$ is $2, 4, 3$, and a different one is $2, 3, 4$ (since they are sequences and so the order matters).

5. there are $P(n,r)$ of these:

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n(n-1)\cdots(n-r+1)\cdots(2)(1)}{(n-r)(n-r-1)\cdots(2)(1)} = \frac{n!}{(n-r)!}.$$

6. Note that both permutations and combinations apply in cases when repetitions are not allowed.

7. Example: The number of 3-digit decimal numbers with no repeated digit is $P(10,3) = 720$ (leading zeros allowed). This could also be done using the product rule: $10{\cdot}9{\cdot}8 = 720$

8. Example: The number of 3-digit decimal numbers with repetition (and leading zeros) allowed is by the product rule $10^3 = 1000$.

9. an <u>r-combination of an n-set A</u> ($r \leq n$) is an r-subset $\{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\}$ of the n-set A. Example: a 3-combination of the 4-set $A = \{1,2,3,4\}$ is $\{2,4,3\}$ (which is the same as the 3-combination $\{2,3,4\}$ since $\{2,4,3\} = \{2,3,4\}$, as the order in a set does not matter).

10. there are $C(n,r)$ of these. The number $C(n,r)$ is also commonly written $\binom{n}{r}$, which is called a *binomial coefficient*. These are associated with a mnemonic called *Pascal's Triangle* and a powerful result called the Binomial Theorem, which makes it simple to compute powers of binomials. (The inductive proof that the binomial theorem is a bit messy, and it becomes easier if it uses the idea of *combinatorial proof*-see MA 3025. A combinatorial proof that we work with here consists of arguing that both sides of an equation of two integer expressions are equal to the cardinality of the same set.)

11. note that we could construct an r-permutation of an n-set in two steps: first take an r-combination, then take a permutation of the r-combination. It follows by the Product Rule that $P(n,r) = r!C(n,r)$, but then

$$C(n,r) = \frac{1}{r!}P(n,r) = \frac{n!}{r!(n-r)!}.$$

This is not a practical formula for hand computation, but we can find a better one without too much difficulty. It looks like this:

$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)(n-r)!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}; \quad (1)$$

note that there are exactly r factors in numerator and denominator alike.

12.
$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-(n-r))!} = C(n,n-r). \qquad (2)$$

This makes some potentially nasty computations pretty easy to carry out. For example, what is $C(100, 98)$? By definition, $C(100, 98) = \frac{100!}{98!2!}$, which is beyond the range of many calculators. And if we use formula (1) for hand computation of r-combinations, we'll have 98 factors in both numerator and denominator. But by (2) and (1) together, we have

$$C(100, 98) = C(100, 2) = \frac{(100)(99)}{2!} = 50 \cdot 99 = 4,950.$$

13. some problems could be solved using both multiplication principle and the permutations. However, the multiplication principle is a good tool for strings where the position of a particular strings is specified (like a string of length 7 that begins with 111), versus the permutations and combinations that apply to the cases where the position is not specified (like a string of length 7 that contains three 1s)